Automorphism Inducing Diffeomorphisms, Invariant Characterization of Homogeneous 3-Spaces and Hamiltonian Dynamics of Bianchi Cosmologies

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Abstract

An invariant description of Bianchi Homogeneous (B.H.) 3-spaces is presented, by considering the action of the Automorphism Group on the configuration space of the real, symmetric, positive definite, 3×3 matrices. Thus, the gauge degrees of freedom are removed and the remaining (gauge invariant) degrees, are the –up to 3– curvature invariants. An apparent discrepancy between this Kinematics and the Quantum Hamiltonian Dynamics of the lower Class A Bianchi Types, occurs due to the existence of the Outer Automorphism Subgroup. This discrepancy is satisfactorily removed by exploiting the quantum version of some classical integrals of motion (conditional symmetries) which are recognized as corresponding to the Outer Automorphisms.

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1 Introduction

In a preceding work [1] we have shown how the presence of the linear constraints, entails a reduction of the degrees of freedom for the quantum theory of Class A spatially homogeneous geometries: the initial six-dimensional configuration space spanned by $\gamma_{\alpha\beta}$'s (the components of the spatial metric with respect to the invariant basis one-forms), is reduced to a space parameterized by the independent solutions to the linear quantum constraints (Kuchař's physical variables [2]). For Bianchi Types $VI_0, VII_0, VIII, IX$ these solutions are the three combinations:

$$x^1 = C^\alpha_{\mu\kappa} C^\beta_{\nu\lambda} \gamma_{\alpha\beta} \gamma^{\mu\nu} \gamma^{\kappa\lambda} \qquad x^2 = C^\alpha_{\beta\kappa} C^\beta_{\alpha\lambda} \gamma^{\kappa\lambda} \qquad x^3 = \gamma$$

(or any other three, independent, functions thereof) and the Wheeler-DeWitt equation becomes a P.D.E. in terms of these x^i's. The Bianchi Type I, where all structure constants are zero (and thus the linear constraints vanish identically), has been exhaustively treated [3]. The Type II case, where only two linear constraints are independent, has been examined along the above lines in [4] and differently in [5].

The fact that the quantum theory (within each one of the above mentioned Bianchi Types) forces us to consider as equivalent any two points $\gamma^{(1)}_{\alpha\beta}, \gamma^{(2)}_{\alpha\beta}$ in the configuration space if they form the same triplet (x^i), seems quite intriguing. It is the purpose of the present work, to investigate in detail the reasons for this grouping of the $\gamma_{\alpha\beta}$'s.
The paper is organized as follows:

Section 2 begins with a careful examination of the action of the general coordinate transformations group on $\gamma_{\alpha\beta}$. The demand that the diffeomorphisms must preserve the manifest homogeneity of the 3-spaces, singles out a particular set of those transformations which has a well defined, non trivial action on $\gamma_{\alpha\beta}$; this action is then proven to be nothing but the action of the automorphism group corresponding to each arbitrary but given Bianchi Type.
The differential description of these automorphic motions, is achieved by identifying the vector fields on the configuration space which, through their integral curves, induce these motions. The importance of Automorphisms in the theory of Bianchi Type Cosmologies, has been stressed in [6].
Concluding this section, we prove the following: if (within a particular albeit arbitrary Bianchi Type) two points $\gamma^{(1)}_{\alpha\beta}, \gamma^{(2)}_{\alpha\beta}$ laying in the configuration space, correspond to the same scalar combinations of $C^\alpha_{\mu\nu}$ and $\gamma_{\alpha\beta}$, then $\gamma^{(2)}_{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \gamma^{(1)}_{\mu\nu}$ where Λ is an element of the corresponding Automorphism group.

In section 3 we briefly recapitulate the essential features of the quantum theory developed in [1], and we compare the purely kinematical results of the previous section, with the ensuing Quantum Hamiltonian dynamics.
For the lower Bianchi Types, this comparison reveals an apparent mismatch between the dynamics and the kinematics. The gap is bridged through the notion of conditional symmetries [7], i.e. some linear in momenta, integrals of motion; their quantum counterparts constrain Ψ to be a function of the geometry only.

Finally, some concluding remarks are included in the discussion.

2 Automorphism Inducing Diffeomorphisms

In this section we shall first relate the action of the Automorphism group on $\gamma_{\alpha\beta}$, to the action induced on it by the class of General Coordinate Transformations (G.C.T.'s) which are subject to the restriction of preservation manifest spatial homogeneity. To this end, consider the spatial line element:

$$ds^2 = \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j \tag{2.1}$$

where $\sigma^\alpha_i(x)dx^i$ are the invariant basis 1-forms, of some given Bianchi Type.

The spatial homogeneity of this line element, is of course, preserved under any G.C.T. of the form:

$$x^i \longrightarrow \widetilde{x}^i = f^i(x) \tag{2.2}$$

Under such a transformation, ds^2 simply becomes:

$$(ds^2 \equiv)d\widetilde{s}^2 = \gamma_{\alpha\beta}(t)\widetilde{\sigma}^\alpha_m(\widetilde{x})\widetilde{\sigma}^\beta_n(\widetilde{x})d\widetilde{x}^m d\widetilde{x}^n \tag{2.3}$$

where the basis one-forms are supposed to transform in the usual way:

$$\widetilde{\sigma}^\alpha_m(\widetilde{x}) = \sigma^\alpha_i(x)\frac{\partial x^i}{\partial \widetilde{x}^m} \tag{2.4}$$

If one were to stop at this point, then one might have concluded that all spatial diffeomorphisms, act trivially on $\gamma_{\alpha\beta}$ i.e. $\gamma_{\alpha\beta} \longrightarrow \widetilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}$. But as we shall immediately see, there are special G.C.T.'s which induce a well-defined, non-trivial action on $\gamma_{\alpha\beta}$. To uncover them, let us ask what is the change in form induced, by tranformation (2.2), to the line element (2.1). To find this change we have to express the line element (2.3) in terms of the old basis one-forms (at the new point) $\sigma^\alpha_i(\widetilde{x})$. There is always a non singular matrix $\Lambda^\alpha_\beta(\widetilde{x})$ connecting $\widetilde{\sigma}$ and σ i.e.:

$$\widetilde{\sigma}^\alpha_m(\widetilde{x}) = \Lambda^\alpha_\mu(\widetilde{x})\sigma^\mu_m(\widetilde{x}) \tag{2.5}$$

Using this matrix Λ we can write line element (2.3) in the form:

$$d\widetilde{s}^2 = \gamma_{\alpha\beta}(t)\Lambda^\alpha_\mu(\widetilde{x})\Lambda^\beta_\nu(\widetilde{x})\sigma^\mu_m(\widetilde{x})\sigma^\nu_n(\widetilde{x})d\widetilde{x}^m d\widetilde{x}^n \tag{2.6}$$

If the functions f^i, defining the transformation, are such that the matrix Λ^α_μ does not depend on the spatial point, then there is a well defined, non trivial action of these transformations on $\gamma_{\alpha\beta}$:

$$\gamma_{\alpha\beta} \longrightarrow \widetilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu\Lambda^\beta_\nu\gamma_{\alpha\beta} \tag{2.7}$$

With the use of (2.4) and (2.5), the requirement that Λ^α_μ does not depend on the spatial point $\widetilde{x}^i$, places the following differential restrictions on the f^is:

$$\frac{\partial f^i(x)}{\partial x^j} = \sigma^i_\alpha(f)S^\alpha_\beta\sigma^\beta_j(x) \tag{2.8}$$

where σ^i_α and S^α_β are the matrices inverse to $\sigma^\alpha_i(x)$ and Λ^α_β, respectively. These conditions constitute a set of first order, highly non-linear P.D.E.'s in the unknown functions f^i. The existence of solutions to these equations, is guaranteed by the Frobenius theorem [8], as long as the necessary and sufficient conditions $\partial^2_{k,\ l} f^i - \partial^2_{l,\ k} f^i = 0$ hold. Through the use of (2.8) and the defining property of the invariant basis 1-forms (3.2), we can transform these conditions into the form:

$$2\sigma^i_\alpha(f)\sigma^\epsilon_k(x)\sigma^\delta_l(x)\left(C^\rho_{\epsilon\delta}S^\alpha_\rho - C^\alpha_{\mu\nu}S^\mu_\epsilon S^\nu_\delta\right) = 0 \tag{2.9}$$

which is satisfied, if and only if, S^α_μ (and thus also Λ^α_μ) is a Lie Algebra Automorphism (see 2.15 bellow). It is, therefore, appropriate to call the General Coordinate Transformations (2.2), when the f^i's satisfy (2.8), *Automorphism Inducing Diffeomorphisms* (A.I.D.'s). The existence of such spatial coordinate transformations is not entirely unexpected: in the particular case $\Lambda^\alpha_\beta(\widetilde{x}) = \delta^\alpha_\beta$ these coordinate transformations, are nothing but the finite motions induced on the hypersurface, by the three Killing vector fields (existing by virtue of homogeneity of the space), which leave the basis one-forms form invariant. The new thing we learn, is that there are further motions leaving the basis one-forms quasi-invariant i.e. invariant modulo a global (space independent) linear mixing, with the mixing matrix Λ^α_μ, belonging to the Automorphism Group. The notion of such transformations ''leaving the invariant triads unchanged modulo a global rotation'' also appears in Ashtekar's work [3], under the terminology ''Homogeneity Preserving Diffeomorphisms''; also the term global is there used in the topological sense.

In order to gain a deeper understanding of the implications of the above analysis as well as the consequences of the kinematics on the dynamics, we have to carefully consider the configuration space and the differential description of the changes (2.7) induced on it, by the A.I.D.'s.

Let us begin with some propositions about the space of 3×3 real, symmetric, (positive definite) matrices:

Proposition 1. *The set Σ of all 3×3 real, symmetric, matrices, forms a vector subspace of $GL(3,\Re)$, and is thus endowed with the structure of a six-dimensional manifold.*

Proposition 2. *The set Δ of all 3×3 real, symmetric, positive definite, matrices, is an open subset of Σ.*

Proof. Let $\gamma_{\alpha\beta}$, be a positive definite 3×3 real, symmetric, matrix, and

$$p(s) = s^3 - As^2 + Bs - C$$

its characteristic polynomial with A, B, C, continuous, polynomial, functions of $\gamma_{\alpha\beta}$'s. Since $\gamma_{\alpha\beta}$ is symmetric, the necessary and sufficient condition that $\gamma_{\alpha\beta}$ be positive definite, is $A > 0, B > 0, C > 0$. Therefore Δ, as an inverse image of an open subset, is itself open. q.e.d.

Proposition 3. *The set Δ is an arcwise connected subset of Σ.*

Proof. Let $\gamma_{\alpha\beta} \in \Delta$. Then, there is $P \in SO(3)$ such that (in matrix notation):

$$P\gamma P^T = D = \mathrm{diag}(a, b, c),$$

with a, b, c the three positive eigenvalues of γ. Since P belongs to $SO(3)$, there is a continuous mapping $\omega : [0,1] \to SO(3)$ such that $\omega(0) = P$ and $\omega(1) = I_3$. Introduce now the mapping $f : [0,1] \to \Delta$, with $f(\sigma) = \omega(\sigma)\gamma\omega(\sigma)^T$. As $\omega(\sigma)$ belongs to $SO(3)$, its determinant is not zero for every $\sigma \in [0,1]$. Therefore, by Sylvester's theorem, $f(\sigma)$ is positive definite –just like γ. But $f(0) = D$ and $f(1) = \gamma$, i.e. the matrix γ is connected to D, by a continuous curve lying entirely in Δ. Consider now the mapping:

$$\phi : [0,1] \to \Delta$$

with:

$$\phi(\sigma) = \mathrm{diag}((a-1)\sigma + 1, (b-1)\sigma + 1, (c-1)\sigma + 1)$$

ϕ is continuous and $\phi(\sigma) \in \Delta$, $\forall\, \sigma \in [0,1]$. This means that γ is finally arcwise connected to I_3. q.e.d.

Let us now proceed with the differential description of motions (2.7). To this end, consider the following linear vector fields defined on Δ:

$$X_{(i)} = \lambda^{\alpha}_{(i)\rho}\gamma_{\alpha\beta}\partial^{\beta\rho} \tag{2.10}$$

with an obvious notation for the derivative with respect to $\gamma_{\alpha\beta}$.
The matrices $\lambda^{\beta}_{(i)\alpha} \equiv (C^{\beta}_{(\rho)\alpha}, \varepsilon^{\beta}_{(i)\alpha})$ are the generators of (the connected to the identity component of) the Automorphism group (see (2.16)) and (i) labels the different generators. Depending on the particular Bianchi Type, the vector fields (in Δ) $X_{(i)}$ may also include, except of the quantum linear constraints (generators of Inner Automorphic Motions) $H_\rho = C^{\alpha}_{\rho\beta}\gamma_{\alpha\kappa}\frac{\partial}{\partial\gamma_{\beta\kappa}}$, the generators of the outer-automorphic motions: $E_{(j)} \equiv \varepsilon^{\sigma}_{(j)\rho}\gamma_{\sigma\tau}\frac{\partial}{\partial\gamma_{\rho\tau}}$

The infinitesimal action of the generic vector field (2.10) $\varepsilon^{(i)}X_{(i)}$ on $\gamma_{\alpha\beta}$ is given by:

$$\bar{\delta}\gamma_{\alpha\beta} \equiv \varepsilon^{(i)}\frac{1}{2}(\lambda^{\mu}_{(i)\alpha}\gamma_{\mu\beta} + \lambda^{\mu}_{(i)\beta}\gamma_{\mu\alpha}) \tag{2.11}$$

where $\varepsilon^{(i)}$ are infinitesimal arbitrary parameters. If we now define the matrices:

$$M^{\mu}_{\alpha} = \varepsilon^{(i)}\lambda^{\mu}_{(i)\alpha} \tag{2.12}$$

we can prove that these, are generators of automorphisms. To see it, let us briefly recall the notion of a Lie Algebra Automorphism: if $\mathcal{A}$ denotes the space of third rank (1,2)

tensors under $GL(3,\Re)$, antisymmetric in the two covariant indices, then the structure constants transform (as it can be inferred from (3.2)) according to:

$$C^{\alpha}_{\mu\nu} \rightarrow \widetilde{C}^{\alpha}_{\mu\nu} = S^{\alpha}_{\beta}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu}C^{\beta}_{\kappa\lambda} \tag{2.13}$$

with Λ^{α}_{μ} and $S^{\alpha}_{\mu} = (\Lambda^{-1})^{\alpha}_{\mu} \in GL(3,\Re)$. A transformation is called a Lie Algebra Automorphism, if and only if, it leaves the structure constants unchanged i.e. if:

$$C^{\alpha}_{\mu\nu} = S^{\alpha}_{\beta}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu}C^{\beta}_{\kappa\lambda} \tag{2.14}$$

or equivalently:

$$C^{\rho}_{\mu\nu}\Lambda^{\alpha}_{\rho} = \Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu}C^{\alpha}_{\kappa\lambda} \tag{2.15}$$

To find the defining relation for the generators λ^{α}_{μ} of the automorphisms Λ^{α}_{μ}, consider a path through the identity $\Lambda^{\rho}_{\theta}(\tau)$, with $\Lambda^{\rho}_{\theta}(0) = \delta^{\rho}_{\theta}$ (we are concerned only with the connected to the identity component of the automorphism group). Differentiating both sides of (2.15) with respect to the parameter τ and setting $\tau = 0$, we get the relation:

$$\lambda^{\alpha}_{\beta}C^{\beta}_{\mu\nu} = \lambda^{\rho}_{\mu}C^{\alpha}_{\rho\nu} + \lambda^{\rho}_{\nu}C^{\alpha}_{\mu\rho} \tag{2.16}$$

where we have identified λ^{α}_{μ}'s with the vectors tangent to the path, at the identity.

By virtue of the Jacobi Identities, one can see that a solution to the system (2.16) is: $\lambda^{\alpha}_{(\kappa)\beta} = C^{\alpha}_{(\kappa)\beta}$ and thus, the structure constants matrices are the generators of the *Inner Automorphisms* proper invariant subgroup of $Aut(G)$. For Bianchi Types $VIII, IX$ these, are the only generators of automorphisms. For all other Bianchi Types, there exist extra matrices satisfying (2.16) –say $\varepsilon^{\alpha}_{(i)\beta}$– generating the *Outer Automorphisms* subgroup of $Aut(G)$. We are now ready to find the finite motions induced on Δ, by the generic vector field $X \equiv \varepsilon^{(i)}X_{(i)}$:

Proposition 4. *Let $\gamma^{(0)}_{\alpha\beta}$ be a fixed point in Δ. Then the curve $\gamma : \Re \rightarrow \Delta$ with:*

$$\gamma_{\alpha\beta}(\tau) = (exp(\tau M))^{\mu}_{\alpha}(exp(\tau M))^{\nu}_{\beta}\gamma^{(0)}_{\mu\nu}$$

is an integral curve (passing through $\gamma^{(0)}_{\alpha\beta}$) of the vector field $X \equiv \varepsilon^{(i)}X_{(i)}$.

Proof. We give a rigorous proof of the statement that the matrices $(exp(\tau M))^{\mu}_{\alpha}$, are automorphisms. To this end, define the mapping $\phi_\tau : \mathcal{A} \rightarrow \mathcal{A}$, with $\phi_\tau(C) = \widetilde{C}$ where $\widetilde{C}^{\alpha}_{\mu\nu} = S^{\alpha}_{\beta}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu}C^{\beta}_{\kappa\lambda}$. Define also the matrices $\Lambda(\tau) = exp(\tau M)$, $S(\tau) = exp(-\tau M)$, with M, given by (2.12). It is straightforward to verify that $\phi_\tau \circ \phi_\sigma = \phi_{\tau+\sigma}$. Using the Jacobi Identities and the definitions above, it is not difficult to see that:

$$\frac{d\phi_\tau(C)}{d\tau}|_{\tau=0} = 0$$

Consider now two sets $\widetilde{C}, C \in \mathcal{A}$, such that $\phi_\psi(C) = \widetilde{C}$, for some ψ. Since the derivative of ϕ_θ at 0 is zero, we have that:

$$\frac{d\phi_\theta(\widetilde{C})}{d\theta}|_{\theta=0} = \lim_{\theta\to 0} \frac{\phi_\theta(\widetilde{C}) - \phi_0(\widetilde{C})}{\theta} = 0$$

which in turn, implies that:

$$\lim_{\theta\to 0} \frac{\phi_\theta \circ \phi_\psi(C) - \phi_\psi(C)}{\theta} = 0 \Rightarrow \lim_{\theta\to 0} \frac{\phi_{\theta+\psi}(C) - \phi_\psi(C)}{\theta} = 0$$

The last expression says that:

$$\frac{d\phi_\psi(C)}{d\psi} = 0, \quad \forall\ \psi$$

i.e. the mapping $\phi_\psi(C)$ is constant $\forall\ \psi$. Thus it holds, in particular, that $\phi_\psi(C) = \phi_0(C)$ or $\widetilde{C}^\alpha_{\mu\nu} = C^\alpha_{\mu\nu}$. q.e.d.

We have thus proven, that the finite motions induced on Δ by $X_{(i)}$ (through its integral curves), are linear transformations of $\gamma_{\alpha\beta}$, of the form (2.7) with $\Lambda \in Aut(G)$. In particular, it is deduced that the linear constraint vector fields generate inner automorphic motions (see [6, 3]).

We now turn out attention, to the invariant description of Bianchi Homogeneous (B.H.) 3-Geometries. It is known that a geometry is invariantly characterized by all its metric invariants. In 3 dimensions all metric invariants, are higher derivative curvature invariants [9], and homogeneity reduces any higher derivative curvature invariant, to a scalar combination of $C^\lambda_{\alpha\beta}$, $\gamma_{\mu\nu}$ –with the appropriate number of C's. So, it is natural to expect that these scalar combinations, will invariantly describe a B.H. 3-geometry. Indeed, it is straightforward to check that any given scalar combination of $C^\lambda_{\alpha\beta}$, $\gamma_{\mu\nu}$, is annihilated by all $X_{(i)}$ defined in (2.10). This, in turn, implies that any such scalar combination is constant (as a function of $\gamma_{\mu\nu}$), along the integral curves of the $X_{(i)}$'s. This fact on account of proposition 4, points to the following

Statement:

Any two hexads $\overset{(2)}{\gamma}_{\alpha\beta}$, $\overset{(1)}{\gamma}_{\alpha\beta}$, for which <u>all scalar</u> combinations of $C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}$ coincide, are automorphically related, i.e. (2.7) holds with $\Lambda \in Aut(G)$.

In order to proceed with the proof, and for latter use as well, it is necessary to define the following scalar combinations of $C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}$ –which constitute a base in the space of all scalar contractions:

$$q^1(C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}) = \frac{m^{\alpha\beta}\gamma_{\alpha\beta}}{\sqrt{\gamma}} \tag{2.17a}$$

$$q^2(C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}) = \frac{(m^{\alpha\beta}\gamma_{\alpha\beta})^2}{2\gamma} - \frac{1}{4} C^\alpha_{\mu\kappa} C^\beta_{\nu\lambda} \gamma_{\alpha\beta} \gamma^{\mu\nu} \gamma^{\kappa\lambda} \tag{2.17b}$$

$$q^3(C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}) = \frac{m}{\sqrt{\gamma}} \tag{2.17c}$$

where $m^{\alpha\beta}$ is the symmetric second rank contravariant tensor density (under the action of $GL(3,\Re)$ in which the structure constants are uniquely decomposed), and m its determinant i.e.:

$$C^{\alpha}_{\beta\gamma} = m^{\alpha\delta}\varepsilon_{\delta\beta\gamma} + \nu_{\beta}\delta^{\alpha}_{\gamma} - \nu_{\gamma}\delta^{\alpha}_{\beta} \tag{2.18}$$

with $\nu_{\alpha} = \frac{1}{2}C^{\rho}_{\alpha\rho}$. At this point the following –easily provable– elements, must be underlined:

E_1 Concerning the number of scalar combinations: the number of independent $\gamma_{\alpha\beta}$'s in a d dimensional space, is $N_1 = d^2 - \binom{d}{2} = d(d+1)/2$ –due to its symmetry. Initially, the number of independent structure constants, is $N_2 = d\binom{d}{2} = d^2(d-1)/2$ –due to the antisymmetry in its lower indices. Taking into account the number of independent Jacobi identities, which is $\binom{d}{2}(d-2) = (d-2)(d-1)d/2$, one is left with $N_3 = N_2 - (d-2)(d-1)d/2 = (d-1)d$ independent structure constants. But, there is also the freedom of arbitrarily choosing $N_4 = d^2$ parameters by linear mixing, i.e. the action of the $GL(3,\Re)$. Thus, the number of independent scalars, which one may construct from the $\gamma_{\alpha\beta}$'s and the $C^{\alpha}_{\mu\nu}$'s, is: $N_s \equiv N_1 + N_3 - N_4 = (d-1)d/2$. For $d = 3$, $N_s = 3$; note that 3 is the maximum number which is achieved only for Bianchi Type $VIII$, IX. In all others, $m = 0$ and, as it can be seen either by direct calculation or from the appendix of [1], the independent scalars are less than 3; namely it is two for Type VI, VII, IV one for Type II, V and 0 for Type I. In each case, the number of the independent q^i's equals the number of curvature invariants.

E_2 The q^i's constitute a complete set of solutions to the system of equations $X_{(i)}\Psi = 0$ i.e. $\Psi = \Psi(q^i)$ is the most general solution to these equations. Since the linear constraint vector fields, H_{α} are in general a subset of the $X_{(i)}$'s, it can be inferred that the q^i's, are solutions to the quantum linear constraints. Except for Type $VIII$, IX, where there are not extra generators, the independent solutions to the quantum linear constraints, include $\gamma = |\gamma_{\alpha\beta}|$ as well as other non scalar combinations [4, 10]. This signals an apparent discrepancy between the kinematics of B.H. 3-spaces previously described, and the quantum dynamics of the (lower) Class A Bianchi Cosmologies.

Now, to resume the line of thought for the proof of the statement, let us define the action of $GL(3,\Re)$ on Δ and $\mathcal{A}$. If $\Lambda^{\alpha}_{\mu},\ S^{\alpha}_{\mu} = (\Lambda^{-1})^{\alpha}_{\mu} \in GL(3,\Re)$ then:

$$\widetilde{\gamma} = \phi_{\Lambda}(\gamma) \stackrel{\text{def}}{\longleftrightarrow} \widetilde{\gamma}_{\alpha\beta} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\gamma_{\mu\nu} \tag{2.19a}$$

$$\widetilde{C} = \phi_{\Lambda}(C) \stackrel{\text{def}}{\longleftrightarrow} \widetilde{C}^{\alpha}_{\mu\nu} = S^{\alpha}_{\beta}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu}C^{\beta}_{\kappa\lambda} \tag{2.19b}$$

As it can be easily inferred from (2.18), $\widetilde{C} = \phi_{\Lambda}(C) \Longrightarrow \widetilde{m}^{\alpha\beta} = |S|^{-1}S^{\alpha}_{\kappa}S^{\beta}_{\lambda}m^{\kappa\lambda}$ and $\widetilde{\nu}_{\alpha} = \Lambda^{\beta}_{\alpha}\nu_{\beta}$. It also holds that $\phi_{\Lambda_1}\circ\phi_{\Lambda_2} = \phi_{\Lambda_2\Lambda_1}$ and obviously, the q^i's in (2.17) satisfy the relation:

$$q^i(\gamma, C) = q^i(\phi_{\Lambda}(\gamma), \phi_{\Lambda}(C)) \tag{2.20}$$

This has the important implication that, when $\Lambda^{\alpha}_{\mu} \in Aut(G)$, the form invariance of the q^i's is guarantied by their explicit definition as scalar combinations of $\gamma_{\alpha\beta}$'s and $C^{\alpha}_{\mu\nu}$'s. The following proposition holds:

Proposition 5. *Let* $\gamma^{(1)}_{\alpha\beta}, \gamma^{(2)}_{\alpha\beta}, \in \Delta$, *and* $C \in \mathcal{A}$ *be the structure constants of a given Bianchi Type. If* $q^i(\gamma^{(1)}, C) = q^i(\gamma^{(2)}, C)$ $(i = 1, 2, 3)$, *then there is* Λ^{α}_{μ} *such that* $\gamma^{(2)} = \phi_{\Lambda}(\gamma^{(1)})$ *and* $\Lambda^{\alpha}_{\mu} \in Aut(G)$ *i.e.* $C = \phi_{\Lambda}(C)$.

To prove this we need the following

Lemma. *If* $q^i(I_3, C_{(1)}) = q^i(I_3, C_{(2)})$ $(i = 1, 2, 3)$ *where* $C_{(1)}, C_{(2)} \in \mathcal{A}$ *are two sets of structure constants corresponding to the same Bianchi Type and* I_3 *is the Identity* 3×3 *matrix, then there exists a matrix* $R \in SO(3)$ *such that* $\phi_R(C_{(1)}) = C_{(2)}$.

Proof of the Lemma. We first note that the number of independent relations in Lemma's hypothesis, equals the number of independent q^i's and is therefore, at most 3. We second observe that in Class A Bianchi Types, the structure constants are characterized by the matrix $m^{\alpha\beta}$ only, and thus the relevant numbers involved are the (at most 3) real, non zero, eigenvalues of $m^{\alpha\beta}$. In Bianchi Type $VIII$ and IX, the non vanishing eigenvalues, are exactly 3. In conclusion, in each and every Class A Bianchi Type, the number of independent relations in Lemma's Hypothesis, exactly equals the number of the non vanishing eigenvalues of matrix $m^{\alpha\beta}$.

In Class B, the null eigenvector ν_{α} of $m^{\alpha\beta}$, is also present. In this case, q^3 vanishes identically, since rank(m) is less than 3 and the number of independent relations in Lemma's Hypothesis is reduced to at most 2. An apparent complication, is thus emerging for Class B Type VI and VII, where the independent relations are two while the relevant numbers are 3 (the two real, non zero, eigenvalues of $m^{\alpha\beta}$ plus the non vanishing component of ν_{α}).
The resolution to this apparent complication, is provided by the algebraic invariant:

$$\lambda \equiv \frac{C^{\tau}_{\tau\mu} C^{\chi}_{\chi\nu} I^{\mu\nu}_3}{C^{\tau}_{\chi\mu} C^{\chi}_{\tau\nu} I^{\mu\nu}_3}$$

This quantity, which is not meant to replace the dynamical variable q^3, vanishes identically in Class A models, while in Class B provides the third relation needed (see [1]).

Thus in every Bianchi Type, 6 numbers appear: in Class A, the 3 eigenvalues of $m^{\alpha\beta}_{(1)}$ which correspond to $C_{(1)}$, and the 3 eigenvalues of $m^{\alpha\beta}_{(2)}$ which correspond to $C_{(2)}$. Similarly, in Class B, the at most 2 eigenvalues of $m^{\alpha\beta}_{(1)}$ plus the third component of its null eigenvector which correspond to $C_{(1)}$ and the at most 2 eigenvalues of $m^{\alpha\beta}_{(2)}$ plus the third component of its null eigenvector which correspond to $C_{(2)}$. The justification for considering only these two triplets and not –for example– the non diagonal components of $m^{\alpha\beta}$, lies in the fact that $m^{\alpha\beta}$ can be put in diagonal form through the action of $SO(3)$, while ν_{α} will have the proper form, for being null eigenvector of $m^{\alpha\beta}$.
So, taking this irreducible form for both the matrix and its null eigenvector, we have the

following relations:
In Class A:

$$\begin{aligned} q^1(I_3, C_1) = q^1(I_3, C_2) \\ q^2(I_3, C_1) = q^2(I_3, C_2) \\ q^3(I_3, C_1) = q^3(I_3, C_2) \end{aligned}$$

while in Class B:

$$\begin{aligned} q^1(I_3, C_1) = q^1(I_3, C_2) \\ q^2(I_3, C_1) = q^2(I_3, C_2) \\ \lambda(I_3, C_1) = \lambda(I_3, C_2) \end{aligned}$$

In each and every case, the corresponding system, can be easily solved, resulting in the equality between the eigenvalues of $m^{\alpha\beta}_{(1)}$ and $m^{\alpha\beta}_{(2)}$, as well as $\nu_{\alpha(1)}$ and $\nu_{\alpha(2)}$. There is thus, a matrix $R \in SO(3)$, such that (in matrix notation) $m_{(2)} = |R|^{-1} R\, m_{(1)}\, R^T$ and $\nu_{(2)} = (R^{-1})^T \nu_{(1)} \Longleftrightarrow C_{(2)} = \phi_R(C_{(1)})$ Of course, $|R| = 1$ and is there, only as a reminder of the tensor density character of $m^{\alpha\beta}$. q.e.d.

Proof of the Preposition 5. Since the matrices $\gamma^{(1)}, \gamma^{(2)}$ are positive definite, there are $\Lambda_{(1)}, \Lambda_{(2)} \in GL(3, \Re)$ such that $\gamma^{(1)} = \phi_{\Lambda_{(1)}}(I_3)$, $\gamma^{(2)} = \phi_{\Lambda_{(2)}}(I_3)$. Let $C_{(1)}, C_{(2)}$ be defined as $C_{(1)} = \phi_{\Lambda^{-1}_{(1)}}(C) \Longleftrightarrow C = \phi_{\Lambda_{(1)}}(C_{(1)})$ and $C_{(2)} = \phi_{\Lambda^{-1}_{(2)}}(C) \Longleftrightarrow C = \phi_{\Lambda_{(2)}}(C_{(2)})$. With C representing again a given, albeit arbitrary Bianchi Type. Using the above and (2.20) we have:

$$\begin{aligned} q^i(\gamma^{(1)}, C) = q^i(\phi_{\Lambda_{(1)}}(I_3), \phi_{\Lambda_{(1)}}(C_{(1)})) = q^i(I_3, C_{(1)}) \\ q^i(\gamma^{(2)}, C) = q^i(\phi_{\Lambda_{(2)}}(I_3), \phi_{\Lambda_{(2)}}(C_{(2)})) = q^i(I_3, C_{(2)}) \end{aligned}$$

The hypothesis $q^i(\gamma^1, C) = q^i(\gamma^2, C)$ translates into $q^i(I_3, C_{(1)}) = q^i(I_3, C_{(2)})$ which through the lemma implies that there is $R \in SO(3)$ such that $C_{(2)} = \phi_R(C_{(1)})$. Since $R \in SO(3)$ (in matrix notation):

$$I_3 = \phi_R(I_3) \Longrightarrow \phi^{-1}_{\Lambda_{(2)}}(\gamma^{(2)}) = \phi_R(\phi^{-1}_{\Lambda_{(1)}}(\gamma^{(1)})) \Longrightarrow \gamma^{(2)} = \phi_{\Lambda_{(2)}} \circ \phi_R \circ \phi_{\Lambda^{-1}_{(1)}}(\gamma^{(1)})$$

Similarly, we have:

$$C_{(2)} = \phi_R(C_{(1)}) \Longrightarrow \phi^{-1}_{\Lambda_{(2)}}(C) = \phi_R(\phi^{-1}_{\Lambda_{(1)}}(C) \Longrightarrow C = \phi_{\Lambda_{(2)}} \circ \phi_R \circ \phi_{\Lambda^{-1}_{(1)}}(C)$$

The above imply that the matrix $\Lambda = \Lambda^{-1}_{(1)} R \Lambda_{(2)}$ satisfies: $\gamma^{(2)} = \phi_\Lambda(\gamma^{(1)})$ and $C = \phi_\Lambda(C)$ i.e. $\Lambda \in Aut(G)$. q.e.d.

We have thus completed the proof of the statement that whenever two hexads form the same multiplet (q^i), they are in automorphic correspondence i.e. (in matrix notation):

$$\exists\ \Lambda \in Aut(G): \ \gamma^{(2)} = \Lambda^T \gamma^{(1)} \Lambda$$

3 Automorphisms and the Linear Constraints

We deem it appropriate to begin this section with a short recalling of the main points of the quantum theory developed in [1]: our starting point is the line element describing the most general spatially homogeneous Bianchi type geometry:

$$ds^2 = (-N^2(t) + N_\alpha(t)N^\alpha(t))dt^2 + 2N_\alpha(t)\sigma^\alpha_i(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j \quad (3.1)$$

where σ^α_i, are the invariant basis one-forms of the homogeneous surfaces of simultaneity Σ_t. Lower case Latin indices, are world (tensor) indices and range from 1 to 3, while lower case Greek indices, number the different basis one-forms and take values in the same range. The exterior derivative of any basis one-form (being a two-form), is expressible as a linear combination of any two of them, i.e.:

$$d\sigma^\alpha = C^\alpha_{\beta\gamma}\ \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_{i,\ j} - \sigma^\alpha_{j,\ i} = 2C^\alpha_{\beta\gamma}\ \sigma^\gamma_i\ \sigma^\beta_j \quad (3.2)$$

The coefficients $C^a_{\mu\nu}$ are, in general, functions of the point x. When the space is homogeneous and admits a 3-dimensional isometry group, there exist 3 one-forms such that the C's become independent of x, and are then called structure constants of the corresponding isometry group.

Einstein's Field Equations for metric (3.1), are obtained, only for the class A subgroup [11] (i.e. those spaces with $C^\alpha_{\alpha\beta} = 0$), from the following Hamiltonian:

$$H = N(t)H_0 + N^\alpha(t)H_\alpha \quad (3.3)$$

where:

$$H_0 = \frac{1}{2}\gamma^{-1/2}L_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} + \gamma^{1/2}R \quad (3.4)$$

is the quadratic constraint, with:

$$\begin{aligned} L_{\alpha\beta\mu\nu} &= \gamma_{\alpha\mu}\gamma_{\beta\nu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu} \\ R &= C^\beta_{\lambda\mu}C^\alpha_{\theta\tau}\gamma_{\alpha\beta}\gamma^{\theta\lambda}\gamma^{\tau\mu} + 2C^\alpha_{\beta\delta}C^\delta_{\nu\alpha}\gamma^{\beta\nu} \end{aligned} \quad (3.5)$$

γ being the determinant of $\gamma_{\alpha\beta}$, R being the Ricci scalar of the slice $t =$ const, and:

$$H_\alpha = 4C^\mu_{\alpha\rho}\gamma_{\beta\mu}\pi^{\beta\rho} \quad (3.6)$$

are the linear constraints.

For Bianchi Types $VI_0, VII_0, VIII$ and IX, all three H_α's are independent. Following Kuchař & Hajicek [2], we can quantize the system (3.3) –with the allocations (3.4), (3.5), (3.6)– by writing the operator constraint equations as:

$$\widehat{H}_\alpha\Psi = C^\beta_{\alpha\mu}\gamma_{\beta\nu}\frac{\partial\Psi}{\partial\gamma_{\mu\nu}} = 0 \quad (3.7)$$

$$\widehat{H}_0\Psi = -\frac{1}{2}\left(\Sigma^{ij}\frac{\partial^2\Psi}{\partial x^i x^j} - \Sigma^{ij}\Gamma^k_{ij}\frac{\partial\Psi}{\partial x^k} + \frac{(\mathcal{D}-2)}{4(\mathcal{D}-1)}R_\Sigma + \sqrt{\gamma}R\right) \tag{3.8}$$

where x^i are the independent solutions to (3.7) and:

$$\Sigma^{ij} = \frac{\partial x^i}{\partial\gamma_{\alpha\beta}}\frac{\partial x^j}{\partial\gamma_{\mu\nu}}\gamma^{-1/2}L_{\alpha\beta\mu\nu}$$

is the induced metric on the reduced configuration space. Also, Γ^k_{ij}, R_Σ, are the corresponding Christoffel symbols and Ricci scalar respectively, while $\mathcal{D} = 3$ (for details such as consistency e.t.c. see [1]). The linear equations (3.7) constitute a system of three independent, first order, P.D.E.'s in the six variables $\gamma_{\alpha\beta}$. These equations, by virtue of the first class algebra satisfied by the operator constraints, admit three independent, non-zero, solutions which can be taken to be the combinations:

$$x^1 = C^\alpha_{\mu\kappa}C^\beta_{\nu\lambda}\gamma_{\alpha\beta}\gamma^{\mu\nu}\gamma^{\kappa\lambda} \qquad x^2 = C^\alpha_{\beta\kappa}C^\beta_{\alpha\lambda}\gamma^{\kappa\lambda} \qquad x^3 = \gamma \tag{3.9}$$

or any other three independent functions thereof. These are Kuchař's physical variables, wich solve the linear constraints. Thus, the presence of the linear constraints at the quantum level, implies that the state vector Ψ must be an arbitrary function of the three combinations (3.9) or any three independent functions thereof. This assumption is also compatible with (3.8), which finally becomes a P.D.E. in terms of the x^i's (see (2.11) of [1]).
In Type II, where only two of the three H_α's are independent, yet another combination of $\gamma_{\alpha\beta}$'s (except the three x^i's in (3.9)) solves (3.7) –see first of [4]. In Type I, all six $\gamma_{\alpha\beta}$'s solve the identically vanishing quantum linear constraints.

Let us now compare this theory with the purely kinematical results of the previous section: to this end, first note that q^1, q^2 in (2.17) solve the quantum linear constraints since, as it can be easily verified:

$$q^1 = \varepsilon\sqrt{\frac{x^1 - 2x^2}{2}} \qquad q^2 = -\frac{x^2}{2}$$

where $\varepsilon = \mathrm{sign}(m^{\alpha\beta}\gamma_{\alpha\beta})$ –see appendix of [1].
In Bianchi Type $VIII$, IX the existence of the non vanishing c-number density m permits us to relate x^3 to the scalar $q^3 = m/\sqrt{x^3}$; thus the grouping entailed by the quantum Hamiltonian dynamics, is completely equivalent to that enforced by the Kinematics of B.H. 3-spaces –described in the previous section.
For Type VI_0, VII_0, $q^3 = 0$ (since $m = 0$) and an apparent discrepancy occurs: kinematically q^1, q^2, (or equivalently x^1, x^2) invariantly and irreducibly characterize a B.H. 3-geometry; that is any function of the 3-geometry, must necessarily and exclusively depend on x^1, x^2. On the other hand, the quantum Hamiltonian dynamics emanating from (3.3) allows $x^3 = \gamma$ as a third possible argument of the wave function which is to solve (3.8). The situation is getting worst when coming to the lower Class A Types. In Type II, the single independent scalar q^1, is adequate for characterizing the 3-slice while –as

explained above– the dynamics allows γ plus two more combinations of $\gamma_{\alpha\beta}$'s. In Type I, not a single q^i, survives while all $\gamma_{\alpha\beta}$'s are –in principle– candidates as arguments of the solution to the Wheeler-DeWitt equation (3.8).
The discrepancy is not of merely academic interest. Any possible argument of the wave function other that the q^i's (or three independent functions thereof) is a gauge degree of freedom since it can be affected by an appropriate A.I.D. A satisfactory solution of the puzzle can be achieved through the usage of the existing conditional symmetries of system (3.3). The detailed analysis has been given for Type I in the last of [3], Type II in second of [4], Type VI_0, VII_0 in [12].
In the rest of this section, we give a brief outline of this idea, and present the characteristic example of Type V, where a complete matching between kinematics and Hamiltonian dynamics occurs.

We first observe that the root of the problem lies in the existence of the generators of the outer automorphic motions $E_{(j)} = \varepsilon^{\sigma}_{(j)\rho}\gamma_{\sigma\tau}\frac{\partial}{\partial\gamma_{\rho\tau}}$ among the $X_{(j)}$'s: their classical counterparts $E_{(j)} = \varepsilon^{\sigma}_{(j)\rho}\gamma_{\sigma\tau}\pi^{\rho\sigma}$ are, at first sight, absent from (3.3). As one can easily compute, the Lie Brackets among these and the generators of the inner automorphic motions H_α's, are:

$$\{H_\alpha, H_\beta\} = -\frac{1}{2}C^{\delta}_{\alpha\beta}H_\delta \quad \{E_{(i)}, H_\beta\} = -\frac{1}{2}\lambda^{\delta}_{(i)\beta}H_\delta \quad \{E_{(i)}, E_{(j)}\} = C'^{(k)}_{(i)(j)}E_{(k)} \tag{3.10}$$

where $\{\ ,\ \}$ stands for the Lie Bracket and $C'^{(k)}_{(i)(j)}$ are particular to each Bianchi Type. So, all the quantum analogues of the $X_{(j)}$'s can be consistently imposed on the wave function:

$$X_{(j)}\Psi = 0 \tag{3.11}$$

as kinematics, dictates. Then, Ψ is a function of the q^i's only –see Table.
The classical dynamics of action (3.3), provides us some, linear in momenta, integrals of motion which are either $E_{(j)}$'s themselves or linear combinations of some of them with $\gamma_{\mu\nu}\pi^{\mu\nu}$ (last of [3], second of [4], [12]). Adopting the recipe that these integrals of motion should also be turned into operators annihilating the wave function, we achieve the desired reconciliation between kinematics and Quantum Hamiltonian dynamics. A very interesting feature is that the corresponding constants of motion, are set equal to zero due to the consistency required (Frobenius Theorem). The following general Type V case, is characteristic:

Although Type V is a Class B model, a valid totally scalar Hamiltonian has been found [13], having the form:

$$H^c \equiv N_0 H^c_0 + N^\rho H^0_\rho = N_0\left(\frac{1}{2}\Theta_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} + V\right) + N^\rho C^{\alpha}_{\rho\beta}\gamma_{\alpha\delta}\pi^{\beta\delta} \tag{3.12}$$

where $\Theta_{\mu\nu\rho\sigma}$ and V a 4th-rank contravariant tensor and scalar respectively, constructed out of the structure constants and $\gamma_{\alpha\beta}$'s. When quantized according to Kuchař & Hajicek, this action gives rise to a wave function depending on 3 combinations of the $\gamma_{\alpha\beta}$'s, namely

$\Psi = \Psi(q^2, \frac{\gamma_{11}}{\gamma_{12}}, \frac{\gamma_{12}}{\gamma_{22}})$ [10]. Clearly, the two last arguments are gauge degrees of freedom since –as one can see from the Table– q^2, is the only invariant characterizing the 3-geometry under discussion.
The elimination of these two degrees of freedom, is achieved by considering the quantum analogues of the following three integrals of motion admitted by (3.12) $E_{(j)} = \varepsilon^{\sigma}_{(j)\rho}\gamma_{\sigma\tau}\pi^{\rho\tau}$, with:

$$\varepsilon^{\alpha}_{(1)\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \varepsilon^{\alpha}_{(2)\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \varepsilon^{\alpha}_{(3)\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One can immediately recognize that these matrices are the outer automorphisms of the Type V Lie Algebra. Consequently, the vector fields $E_{(j)} = \varepsilon^{\sigma}_{(j)\rho}\gamma_{\sigma\tau}\frac{\partial}{\partial\gamma_{\rho\tau}}$ are generating outer automorphic motions in the configuration space. Turning these integrals of motion into operators imposed on Ψ, i.e. demanding $E_{(j)}\Psi = M_j\Psi$ and utilizing the algebra (say C'^{k}_{ij}) which the previous three matrices obey, one arrives at the consistency condition $C'^{k}_{ij}M_k = 0$, implying that the constants of integration M_k, should be set equal to zero. We thus retrieve all the conditions $X_{(j)}\Psi = 0$ –required by the kinematics.

So we have an example in which the dynamics completely complies with the kinematical/geometrical results, obtained in section 2. As we have earlier mentioned, the same situation occurs for all Class A Types –when $E_{(j)}$'s exist. In the case of Type $VIII$, IX the Hamiltonian (3.3) is totally scalar since, $m/\sqrt{\gamma}$ is the q^3 –m being a c-number density.

4 Discussion

In section 2, we first identified the particular class of G.C.T.'s which preserve manifest homogeneity of the line element of the generic B.H. 3-space. Their action on the configuration space spanned by $\gamma_{\alpha\beta}$'s, is shown to be that of the Automorphism group. The differential description of this action on Δ, leads us to the vector fields $X_{(j)}$'s. Their characteristic solutions, the q^i's, irreducibly and invariantly label the 3-geometry. Thus for any given but arbitrary Bianchi Type, points in Δ, corresponding to the same multiplet q^i, are automorphically related and thus G.C.T. equivalent. A first conclusion concerning any possible quantum theory of Bianchi Cosmologies, is thus reached on solely kinematical grounds; the wave function must depend on q^i's only –if it is to represent the geometry and not the coordinate system on the 3-slice.

In section 3 we first present the quantization of Hamiltonian action (3.3) according to Kuchař's and Hajiceck's recipe. We see that the quantum linear constraint vector fields H_α's corresponding to the inner automorphisms proper invariant subgroup $InAut(G)$ of $Aut(G)$ are among the $X_{(i)}$'s.
As seen from the table, for Types $VIII$, IX there are no outer-automorphisms and the three x^i's are in one-to-one correspondence to the three q^i's (essentially the three independent curvature invariants). For all other Class A Types, there is always an outer-automorphism with non-vanishing trace; the corresponding generator in configuration

space Δ does not (weakly) commute with the quadratic constraint (3.4) nor does its corresponding quantum analogue commute with (3.8). Thus, for the lower Class A Types, the wave functions emanating from action (3.3), depend on the curvature invariants and on γ despite that $q^3 = 0$; these wave functions will therefore not be G.C.T. invariant, since γ can be changed to anything we like by an A.I.D. This result seems to justify (for these types) the claim made by some authors, that γ should be considered as time variable and thus frozen out [14]. One may say that for the lower Class A Types the grouping dictated by the quantum theory, resulting from action (3.3), is overcomplete: although any two hexads forming the same x^1, x^2 are geometrically identifiable (since they correspond to G.C.T. related spatial line-elements), the theory requires that $x^3 = \gamma$ be also the same in order to consider these two hexads as equivalent. At first sight, this may be seen as a defect of the classical action (3.3); although it reproduces Einstein's Equations for (Class A) spatially homogeneous spacetimes, it does not correctly reflects the full covariances of these equations. However, as is explained in [4, 10, 12, 3], the conditional symmetries of this action rectify this defect: for Bianchi Types other than $VIII$, IX, there are extra, linear in momenta, integrals of motion –say $E_{(i)} = \varepsilon^{\alpha}_{(i)\rho}\gamma_{\alpha\sigma}\pi^{\rho\sigma}$– corresponding to the outer automorphisms subgroup of $Aut(G)$. It is shown how the quantum analogues of these $E_{(i)}$'s can serve to satisfactorily remove this discrepancy. Their imposition as additional conditions restricting the wave function, results in forcing it to depend on q^i's only. A noteworthy feature of this procedure is that, at the quantum level, the consistency requirement of these extra conditions leads to setting zero, the classical constants of integration (which are non essential) –as the particular example of Type V, exhibits.

Another important consequence of the results in section 2, is the conclusion that a Homogeneous 3-Geometries are completely characterized by their curvature invariants: indeed, as it is well known, in 3 dimensions all metric invariants are higher derivative curvature invariants [9]; but the homogeneity of the space reduces any higher derivative curvature invariant to a scalar combination of $C^{\alpha}_{\mu\nu}, \gamma_{\alpha\beta}$ with the appropriate number of C's. Thus any two distinct Homogeneous 3-Geometries must differ by at least one curvature invariant, i.e. by at least one q^i; and vice versa, any two Homogeneous 3-metrics for which all curvature invariants (i.e. all q^i's) coincide, are necessarily G.C.T. related and thus represent the same 3-Geometry.

Last but not least, we would like to underline that the partitioning of the Automorphism Group in Inner and Outer Subgroups, which quantum theory seems to favour, does have a classical analogue: the inner automorphism parameters, represent genuine ''gauge'' degrees of freedom (i.e. can be allowed to be arbitrary functions of time) –see 4th of [6]–, while the outer automorphism parameters, are rigid symmetries –3rd of [6].

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Table

Type	Generators $\lambda^{\alpha}_{(i)\beta}$	# of Indep. Parameters	# of Indep. H_α's	# of Indep. E_α's	# of Indep. q^i's
I	$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix}$	9	0	0	0
II	$\begin{pmatrix} p_3+p_6 & p_1 & p_2 \\ 0 & p_3 & p_4 \\ 0 & p_5 & p_6 \end{pmatrix}$	6	2	3	1
III	$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_1 & p_4 \\ 0 & 0 & 0 \end{pmatrix}$	4	2	2	2
IV	$\begin{pmatrix} p_1 & p_2 & p_3 \\ 0 & p_1 & p_4 \\ 0 & 0 & 0 \end{pmatrix}$	4	3	1	2
V	$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ 0 & 0 & 0 \end{pmatrix}$	6	3	2	1
VI	$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_1 & p_4 \\ 0 & 0 & 0 \end{pmatrix}$	4	3	1	2
VII	$\begin{pmatrix} p_1 & p_2 & p_3 \\ -p_2 & p_1 & p_4 \\ 0 & 0 & 0 \end{pmatrix}$	4	3	1	2
VIII	$\begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & 0 & p_3 \\ p_2 & -p_3 & 0 \end{pmatrix}$	3	3	0	3
XI	$\begin{pmatrix} 0 & p_1 & p_2 \\ -p_1 & 0 & p_3 \\ -p_2 & -p_3 & 0 \end{pmatrix}$	3	3	0	3

Notes:

N_1 *The number of the independent q^i's equals the number of the independent curvature invariants.*

N_2 *Type III, is characterized by the condition $h = \pm 1$, while Type VI, by the condition $h \neq (0, \pm 1)$.*

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